Homework #4

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1 Number 1

Show that the following are context-free:

1. $\{\mathrm{a}^n w \mathrm{c} w^R \mathrm{a}^{5n+4} | n \geq 0; w \in \{a, b\}\}$

$$\begin{aligned} \mathrm{L} &\rightarrow \mathrm{AWaaaa} \quad (1) \\ \mathrm{A} &\rightarrow \mathrm{aAaaaaa}|\lambda \quad (2) \\ \mathrm{W} &\rightarrow \mathrm{aWa}|\mathrm{bWb}|\mathrm{c} \quad (3) \end{aligned}$$

2. $\{\mathrm{a}^i \mathrm{b}^j \mathrm{c}^k \mathrm{d}^l | i, j, k, l \geq 0; i \geq j; k \geq l\}$

$$\begin{aligned} \mathrm{L} &\rightarrow \mathrm{AC} \quad (4) \\ \mathrm{A} &\rightarrow \mathrm{aAb}|\mathrm{aA}|\lambda \quad (5) \\ \mathrm{W} &\rightarrow \mathrm{cCd}|\mathrm{cC}|\lambda \quad (6) \end{aligned}$$

3. $\{\mathrm{a}^n \mathrm{b}^{3n+7} \mathrm{a}^{2m+3} \mathrm{b}^{6m} | n, m \geq 0\}$

$$\begin{aligned} \mathrm{L} &\rightarrow \mathrm{AC} \quad (7) \\ \mathrm{A} &\rightarrow \mathrm{Bbbbbbbb} \quad (8) \\ \mathrm{B} &\rightarrow \mathrm{aBbbb}|\lambda \quad (9) \\ \mathrm{C} &\rightarrow \mathrm{aaaD} \quad (10) \\ \mathrm{D} &\rightarrow \mathrm{aaDbbbbbb}|\lambda \quad (11) \end{aligned}$$

4. $\{a^n b^n | n\%7 \neq 0\}$

$$\begin{aligned}
L &\rightarrow ab|aA_2b && (12)\\
A_2 &\rightarrow ab|aA_3b && (13)\\
A_3 &\rightarrow ab|aA_4b && (14)\\
A_4 &\rightarrow ab|aA_5b && (15)\\
A_5 &\rightarrow ab|aA_6b && (16)\\
A_6 &\rightarrow ab|aA_7b && (17)\\
A_7 &\rightarrow aLb && (18)
\end{aligned}$$

2 Number 2

If T is regular and U is context free, what is true about TU? Why?

TU is context free.

Since λ is regular, TU is trivially not regular.

Since T can be simulated with a DFA and U with a PDA. A DFA is a degenerate case of a PDA where each time nothing is pushed or popped from the stack. To model TU with a PDA simply combine the PDA for T and U by making a λ transition from each element in the final states of T to the start state of U. The resultant PDA will accept TU $\therefore$ TU is context free.

3 Number 3

If T is context free and U is regular, what is true about T - U? Why?

T - U is context free.

Since λ is regular, T - U is trivially not regular.

T - U = $T \cap \overline{U}$. Since U is regular, $\overline{U}$ is regular too. It is possible to create a pda which simulates T and $\overline{U}$ in parallel, accepting $T \cap \overline{U}$ $\therefore$ T - U is context free.

4 Number 4

If X^R is context free, is X context free? Why?

Yes, X is context free if X^R is context free. For a grammar X $\ni$

$$\begin{aligned} \mathrm{X} &= \{P, \Sigma, S, R\} & (19)\\ P &\text{ is the set of nonterminals} \\ \Sigma &\text{ is the input alphabet (terminals)} \\ S &\in P \text{ is the start state} \\ R &\text{ is the set of rules} \\ &= \{(l, r)|l \in P, r \in \{P \cup \Sigma\}^*\} & (20)\end{aligned}$$

There exists a grammar for $X^R = \{P, \Sigma, S, R^R\} \ni$

$$R^R = \{(l_R, r_R)|\forall(l, r) \in R; l_R = l, r_R = r^R\} \tag{21}$$

5 Number 5

Prove that $L = \{a^k|k = 2^m; m \in \mathbb{N}\}$ is not context free.

Pick $w = a^k \ni k = 2^n$. $|w| = 2^n > n$ and $w \in L$, so the pumping lemma holds if L is context free $\therefore \exists w \in L; n \in \mathbb{N} \ni$

$$\begin{aligned} w &= uvxyz & (22)\\ |w| &\geq n & (23)\\ |vxy| &\leq n & (24)\\ |v| + |y| &\geq 1 & (25)\\ w_i &= uv^ixy^iz \in L \forall i \in \mathbb{N} & (26)\\ |w_i| &= |u| + (i)|v| + |x| + (i)|y| + |z| & (27)\end{aligned}$$

Since $1 \leq |v| + |y| \leq n$ from the pumping lemma,

$$2^n = |w_1| < |w_2| = |w_1| + |v| + |y| \leq 2^n + n < 2^{n+1} \tag{28}$$

So, $w_2 \notin L \therefore$ L is not context free.

6 Number 6

Explain why $\mathrm{T} = \{\mathrm{a}^i\mathrm{b}^i\mathrm{c}^j\mathrm{d}^j|i, j \geq 0\}$ is context free.

Because it can be described by the grammar:

$$\begin{aligned} \mathrm{L} &\rightarrow AC \quad (29) \\ \mathrm{A} &\rightarrow aAb|\lambda \quad (30) \\ \mathrm{C} &\rightarrow cCd|\lambda \quad (31) \end{aligned}$$

7 Number 7

Explain why $\mathrm{U} = \{\mathrm{a}^i\mathrm{b}^j\mathrm{c}^j\mathrm{d}^k | i, j, k \geq 0\}$ is context free.

Because it can be described by the grammar:

$$\begin{aligned} \mathrm{L} &\rightarrow ABD \quad (32) \\ \mathrm{A} &\rightarrow aA|\lambda \quad (33) \\ \mathrm{B} &\rightarrow bBc|\lambda \quad (34) \\ \mathrm{D} &\rightarrow dD|\lambda \quad (35) \end{aligned}$$

8 Number 8

Explain why, for T and U from the previous problems, $\mathrm{X} = \mathrm{T} \cap \mathrm{U}$ is not context free.

$$\begin{aligned} \mathrm{T} \cap \mathrm{U} &= \{\mathrm{a}^i\mathrm{b}^j\mathrm{c}^k\mathrm{d}^l | i = j; k = l; j = k\} \quad (36) \\ &= \{\mathrm{a}^i\mathrm{b}^j\mathrm{c}^k\mathrm{d}^l | i = j = k = l\} \quad (37) \end{aligned}$$

First define a homomorphism $h \ni$

$$\begin{aligned} h(\mathrm{a}) &= a \quad (38) \\ h(\mathrm{b}) &= b \quad (39) \\ h(\mathrm{c}) &= c \quad (40) \\ h(\mathrm{d}) &= \lambda \quad (41) \end{aligned}$$

$$\mathrm{X}_h = h(\mathrm{X}) = h(\mathrm{T} \cap \mathrm{U}) = \{\mathrm{a}^i\mathrm{b}^i\mathrm{c}^i | i \geq 0\} \quad (42)$$

$\{\mathrm{a}^i\mathrm{b}^i\mathrm{c}^i | i \in \mathbb{N}\}$ is one of the canonical non-context free grammars.

9 Number 9

Present a formal definition of a two stack push-down automata and a description of its language.

A normal push-down automata is defined as follows:

$$\begin{aligned}
A &= (\Sigma, Q, \Gamma, \delta, q_0, F) && (43)\\
\Sigma &\text{ is } \text{an input alphabet}\\
Q &\text{ is } \text{a set of states}\\
\Gamma &\text{ is } \text{a stack alphabet}\\
\delta &\text{ is } \text{a transition function}\\
\delta &\models Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \mapsto Q \times \Gamma^* && (44)\\
q_0 &\in Q \text{ is a start state}\\
F &\subseteq Q \text{ is a set of final states}
\end{aligned}$$

To add another stack all that needs to be altered is the transition function since the basic idea doesn't change. Define a double stack pda, A_d as:

$$\begin{aligned}
A_d &= (\Sigma, Q, \Gamma, \delta_d, q_0, F) && (45)\\
\delta_d &\models Q \times (\Sigma \cup \{\epsilon\}) \times (\Sigma \cup \{\epsilon\}) \times \Gamma \mapsto Q \times \Gamma^* && (46)
\end{aligned}$$

δ is defined as a function:

$$\begin{aligned}
\delta(q_i, \sigma, \gamma_o) &= (q_j, \gamma_u) && (47)\\
q_i &\in Q \text{ is an initial state}\\
\sigma &\in \Sigma \text{ is an input character}\\
\gamma_o &\in \Gamma \text{ is a character to pop from the stack}\\
q_j &\in Q \text{ is a resultant state}\\
\gamma_u &\in \Gamma \text{ is a character to push on the stack}
\end{aligned}$$

δ_d has the same basic definition as δ with a change in how the stack alphabet is handled:

$$\begin{aligned}
\delta_d(q_i, \sigma, \gamma_{o1}, \gamma_{o2}) &= (q_j, \gamma_{u1}, \gamma_{u2}) && (48)\\
\gamma_{oi} &\ni i \in \{1, 2\} \in \Gamma\\
&\text{is popped from the first and second stack respectively}\\
\gamma_{ui} &\ni i \in \{1, 2\} \in \Gamma\\
&\text{is pushed on the first and second stack respectively}
\end{aligned}$$

10 Number 10

Design a two stack PDA to accept $\mathrm{T} \cup \mathrm{U}$ from the previous problem.

$\mathrm{T} \cup \mathrm{U} = \{\mathrm{a}^i\mathrm{b}^i\mathrm{c}^i\mathrm{d}^i | i \in \mathbb{N}\}$ can be generated by the following two stack pda:

$$\begin{aligned}
A_d &= (\Sigma, Q, \Gamma, \delta_d, q_0, F) && (49)\\
\Sigma &= \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\} && (50)\\
Q &= \{q_0, q_1, q_2, q_3\} && (51)\\
\Gamma &= \{1\} && (52)\\
F &= \{q_3\} && (53)\\
\delta_d(q_0, \mathrm{a}, \lambda, \lambda) &= (q_0, 1, \lambda) && (54)\\
\delta_d(q_1, \mathrm{b}, 1, \lambda) &= (q_1, \lambda, 1) && (55)\\
\delta_d(q_2, \mathrm{c}, \lambda, 1) &= (q_2, 1, \lambda) && (56)\\
\delta_d(q_3, \mathrm{d}, 1, \lambda) &= (q_3, \lambda, \lambda) && (57)\\
\delta_d(q_i, \lambda, \lambda, \lambda) &= (q_{i+1}, \lambda, \lambda) \ni i \in \{0, 1, 2\} && (58)
\end{aligned}$$